

Every Schnyder Drawing is a Greedy Embedding

Pierre Leone and Kasun Samarasinghe

Centre Universitaire d'Informatique

University of Geneva

Switzerland

{pierre.leone,kasun.wijesiriwardana}@unige.ch

Abstract

Geographic routing is a routing paradigm, which uses geographic coordinates of network nodes to determine routes. Greedy routing, the simplest form of geographic routing forwards a packet to the closest neighbor towards the destination. A greedy embedding is an embedding of a graph on a geometric space such that greedy routing always guarantees delivery. A Schnyder drawing is a classical way to draw a planar graph. In this manuscript, we show that every Schnyder drawing is a greedy embedding, based on a generalized definition of greedy routing.

1 Greedy Routing on Planar Triangular Graphs

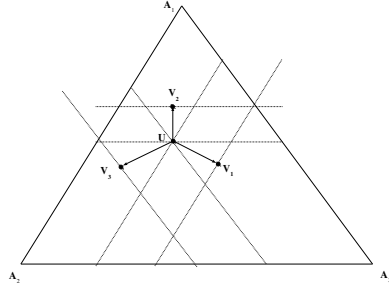
In this manuscript, we establish some results on greedy routing on planar triangular graphs. Planar triangular graphs are a special class of graphs, where every face of the graph is a triangle, including the external face. Hence the outer-face consists of three nodes A_1 , A_2 and A_3 .

We consider the problem of greedy routing [1] on such graphs. In order to perform geometric routing, the graph has to be drawn on a certain geometric space. Such a drawing is called a greedy embedding [2], which is defined as follows.

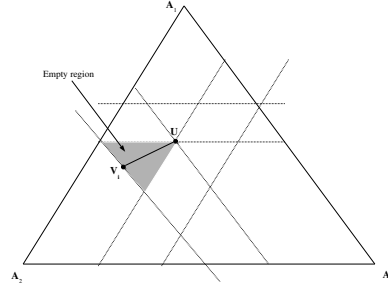
Definition 1. *Greedy Embedding* *A greedy embedding is an embedding of a graph on a respective geometric space such that, greedy routing always succeeds. In other words, between every node pair u, v there is another node w adjacent to u , such that $d(u, v) > d(w, v)$, where $d(\cdot)$ is the underlying metric on the geometric space.*

Dhandapani [3] showed that every planar triangulated graph can be drawn on the plane as a greedy embedding. They generalize the classical Schnyder drawing [4] leading to a family of planar drawings, then they show that there exists a greedy drawing in this set of drawings.

In this work, we prove that every Schnyder drawing is a greedy embedding. We emphasize the use of a generalized definition of a greedy routing [5], on which our algorithm is based.



(a) Illustration of Lemma 1: There are exactly three edges in the three regions S_1^u, S_2^u and S_3^u



(b) Gray region is empty of nodes; Enclosing triangle property (Lemma 6) [3]

Figure 1: Two cases to consider in greedy path construction

2 Schnyder Drawing

Given a planar triangular graph, a Schnyder drawing [4] is a straight line drawing of the graph on the plane. In this article, we consider such a drawing on \mathbb{R}^3 , such that the external nodes A_1, A_2 and A_3 are placed on $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. Hence the external face forms an equilateral triangle and the nodes are placed on the plane designated by $x + y + z = 1$.

The Schnyder drawing is computed based on a combinatorial description of a planar triangular graph, which is called a *realizer*, defined as follows.

Theorem 1. Realizer[4] *Given a plane triangulation $G(V, E)$, there exist three directed edge-disjoint trees, T_1, T_2 and T_3 , namely the realizer of G , such that for each inner vertex u ;*

1. *u has an outgoing edge in each of T_1, T_2 and T_3*
2. *the counterclockwise order of the edges incident on v is as follows: leaving in T_1 , entering in T_3 , leaving in T_2 , entering in T_1 , leaving in T_3 , entering in T_2*

The Schnyder planar drawing algorithm [4], initially constructs a realizer in linear time. A realizer, in turn leads to three paths from each node towards their root nodes in each tree. These paths partition the nodes into three regions R_i such that $i = 1, 2, 3$. Let n_i be the total number of nodes in region R_i including the nodes in the two paths, those border the region R_i . Now Schnyder algorithm places each node u on $\frac{1}{n}(n_1, n_2, n_3)$ leading to a planar drawing (see [6] for details) [4].¹

We present two important properties of a Schnyder drawing in Lemma 1 and 2, which are illustrated in Figures 1a and 1b²

Lemma 1. The Three Wedges Property Lemma 4[3]

In every Schnyder drawing the three outgoing edges at an internal vertex v have

¹Note that the drawing we consider forms an equilateral triangle, while in [6] they present an algorithm where the outer face does not form an equilateral triangle

²See [3] for the proof

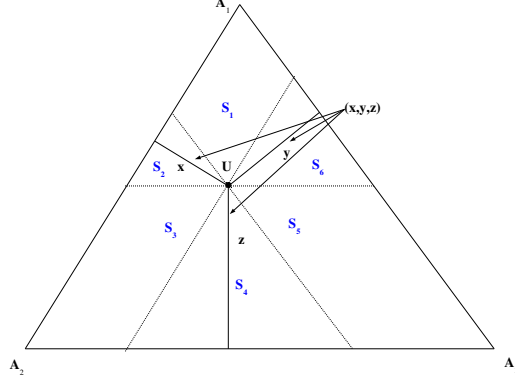


Figure 2: A node estimates the perpendicular distances from the outer-face edges and assign itself the raw distances as the coordinate (x,y,z)

slopes that fall in the intervals P_1 in $[60^\circ, 120^\circ]$, P_2 in $[180^\circ, 240^\circ]$ and P_3 in $[300^\circ, 360^\circ]$, with exactly one edge in each interval as shown in Figure 1a

Schnyder drawing leads to an important void region around and out-going edge as constitutes in Lemma 2

Lemma 2. The enclosing triangle property Lemma 6[3] Given a vertex u and an outgoing edge (u,v) belonging, without loss of generality to P_1 , the equilateral triangle formed by drawing lines with slopes of $0^\circ, 60^\circ$ and 120° , as shown in Figure 1b is free of any other vertices. This results holds for both regions P_2 and P_3 .

In the following section, we introduce the definition of a saturated graph and show that every Schnyder drawing implies a saturated graph.

3 Saturated Graph

In [7], we introduced the concept of saturated graphs and show that there exists a local greedy routing algorithm with guaranteed delivery on such graphs. Definition of a saturated graph is purely combinatorial, but to construct those properties we use an underlying virtual coordinate system namely, virtual raw anchor coordinate (VRAC) system [7]. In the following, we describe the construction of a saturated graph, given a Schnyder drawing of a planar triangular graph.

Let $G(V, E)$ is the graph in concern, where V is the set of vertices(nodes) and E is the set of edges, which is a planar triangular graph. Consider a Schnyder drawing of G , where the outer face is drawn as an equilateral triangle. We denote the three outer vertices as A_1, A_2 and A_3 . Let every node computes the distance from the edges of the outer-triangle and assign them as its coordinate as illustrated in Figure 2, which is in fact the VRAC system. Based on this coordinate assignment, we can define three order relations on the set of nodes V as follows.

Definition 2. The three order relations $<_i$, $i = 1, 2, 3$ on $V \times V$ are defined by

$$\forall u, v \in V \quad u <_i v \iff d(u, A_i) > d(v, A_i) \iff u_i < v_i.$$

These three orders permit the definition of sectors associated with a node u .

Definition 3. We define the following sectors associated with a node $u \in V$. Note that the reference node u does not belong either of to the sectors. As depicted in Figure 2, sectors s_i^u correspond to the labeled regions.

$$\begin{aligned} s_1^u &= \{v \mid u <_1 v, u >_2 v, u >_3 v\} \cap \widehat{A_1 A_2 A_3}. \\ s_2^u &= \{v \mid u <_1 v, u <_2 v, u >_3 v\} \cap \widehat{A_1 A_2 A_3}. \\ s_3^u &= \{v \mid u >_1 v, u <_2 v, u >_3 v\} \cap \widehat{A_1 A_2 A_3}. \\ s_4^u &= \{v \mid u >_1 v, u <_2 v, u <_3 v\} \cap \widehat{A_1 A_2 A_3}. \\ s_5^u &= \{v \mid u >_1 v, u >_2 v, u <_3 v\} \cap \widehat{A_1 A_2 A_3}. \\ s_6^u &= \{v \mid u <_1 v, u >_2 v, u <_3 v\} \cap \widehat{A_1 A_2 A_3}. \end{aligned}$$

Based on this characterization, Leone et.al [8], proposed a distributed planarization algorithm, assuming a unit disk communication graph model. They used the following planarization criterion, where they derive a planar sub-graph $G(V, \tilde{E})$.

Definition 4. The vertex set $\tilde{V} = V$ and

$$\tilde{E} = \left\{ (u, v) \mid v \in s_{2k-1}^u \text{ and } v = \min_k(s_{2k-1}^u) \text{ } k = 1, 2 \text{ or } 3 \text{ and } (u, v) \in E \right\} \quad (1)$$

Algorithmically, to obtain a planar graph, each node retains only the minimum edge in each sector s_1^u, s_3^u and s_5^u . The minimum edge in a given sector is determined based on a partial order corresponds for each sector. Such a partial order is defined considering the intersection of three orders ($<_i$, $i = 1, 2, 3$) in a given sectors (see [8] for details). Note that given an arbitrary graph, these three sectors may not have edges and there can be arbitrary number of incoming edges to a node in sectors s_2^u, s_4^u and s_6^u . A saturated graph is a special case of this setting, which is defined as below.

Definition 5 (Saturated Graph). A planar graph is saturated if there exists exactly one edge in each sector s_{2i-1}^u , $i = 1, 2, 3$ for each node u .

We present the following result from [7] on greedy routing on saturated graphs.

Theorem 2. There is a greedy routing algorithm on every saturated planar graph.

3.1 Schnyder drawings and saturated graphs

A saturated graph is defined without a reference to an embedding. Our greedy routing algorithm in [7], uses the saturated graph property to prove the delivery guarantees. Following lemma constitutes a straight forward relationship between a Schnyder drawing and a saturated graph.

Lemma 3. *Every Schnyder drawing implies a saturated graph.*

Proof. Due to the three wedge property of a Schnyder drawing (see Lemma 1), we know that there is exactly one edge (u, v_i) in sectors s_i^u where $i = 1, 3, 5$. Moreover due to the *enclosing triangle property* (see Lemma 2) of a Schnyder drawing, there is no node w such that $w <_j v$ where $j = 1, 2, 3$. Hence it follows the criterion for a saturated edge as in equation 1, implying a saturated graph. \square

In [7], we devised a greedy routing algorithm which guarantees delivery, when the graph is saturated. Note that in [7], we do not use a metric to define the greedy path, instead use a generalized definition of a greedy path. Following lemma concludes the resulting connection between greedy embeddings and a Schnyder drawing of a planar triangular graph.

Corollary 1. *Every Schnyder drawing is a greedy embedding.*

References

- [1] Prosenjit Bose and Pat Morin. Online routing in triangulations. In *Algorithms and Computation*, pages 113–122. Springer, 1999.
- [2] Christos H Papadimitriou and David Ratajczak. On a conjecture related to geometric routing. In *Algorithmic Aspects of Wireless Sensor Networks*, pages 9–17. Springer, 2004.
- [3] Raghavan Dhandapani. Greedy drawings of triangulations. *Discrete & Computational Geometry*, 43(2):375–392, 2010.
- [4] Walter Schnyder. Embedding planar graphs on the grid. In *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms*, pages 138–148. Society for Industrial and Applied Mathematics, 1990.
- [5] Yujun Li, Yaling Yang, and Xianliang Lu. Rules of designing routing metrics for greedy, face, and combined greedy-face routing. *Mobile Computing, IEEE Transactions on*, 9(4):582–595, 2010.
- [6] Takao Nishizeki and Md Saidur Rahman. *Planar graph drawing*, volume 12. World Scientific, 2004.
- [7] Pierre Leone and Kasun Samarasinghe. Greedy routing on virtual raw anchor coordinate system. In *Distributed Computing in Sensor Systems (DCOSS), 2016 IEEE 12th International Conference on*, 2016.
- [8] Florian Huc, Aubin Jarry, Pierre Leone, and Jose Rolim. Efficient graph planarization in sensor networks and local routing algorithm. In *Distributed Computing in Sensor Systems (DCOSS), 2012 IEEE 8th International Conference on*, pages 140–149. IEEE, 2012.